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Abstract

The operation of autonomous systems is inherently constrained by their surrounding environment, which is often time-varying and unknown *a priori*, necessitating perception using sensors. Hence, control strategies for autonomous systems must take into account the uncertainty of the perceived environment in making decisions, while information acquired by sensors often depends on how the system is operated, e.g., where the sensors are pointed at, or what and how much sensor information is processed. We introduce a perception-aware chance-constrained model predictive control (PAC-MPC) strategy that accounts for the uncertainty of the perceived environment, as well as the dependence of the perception quality on the control actions. The system and the environment are coupled by chance constraints due to the uncertainty in the environment estimate, which depends on control actions. We establish the constraint satisfaction and stability properties of PAC-MPC through appropriate design of the cost function and terminal set, and propose a constructive design procedure for the case of linear dynamics.

Key words: Control of constrained systems under uncertainty, Nonlinear predictive control, Perception and sensing, Integration of control and perception

1 Introduction

Autonomous systems, such as mobile robots, automated vehicles and drones, admit motion models that are generally accurate, especially when regulated by low-level controllers. However, the environment where such systems operate is often not known *a priori* and may change dynamically, e.g., due to the location of other vehicles on the road, of workers on a factory floor, of obstructions to flight path, or of markings delimiting the workspace. The elements of the environment may not affect the system motion dynamics, but rather impose constraints on the permissible motions and actions, such as containment on the workspace, collision avoidance with vehicles or workers, and safe flyby around obstructions. The environment information is obtained from perception using data provided by sensors, e.g., LIDAR, radar and cameras [6]. Since the perception from sensors is imperfect, the constraints relating system and environment are uncertain and hence the perception performance in reducing the environment uncertainty affects the control decisions [1, 11, 17, 18].

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While perception is often assumed to be independent from control actions, that may overlook the actual capabilities of advanced sensors. For instance, cameras and LIDARs have a limited field-of-view and range, and are subject to occlusions, so that the acquired information depends on system position and attitude. Advanced sensors are often equipped with mechanisms that allow different regions and amounts of focus, such as radar beamforming, and hence allow to directly or indirectly adjust the spread and quality of the acquired information. Based on this, the system operation may improve if the controller can predict the impact of control decisions on the uncertainty of the perceived environment information and, accordingly, account for such varying uncertainty in decision-making. For instance, in automated driving, the control stack may determine an initial trajectory that provides a better field of view or avoids occlusions, while also determining the focus areas for radar beamforming. This may reduce the uncertainty in critical road areas, so that more effective trajectories become subsequently feasible. As a complicating factor, advanced sensors in actual applications are equipped with on-board perception algorithms that the controller may leverage but not re-design, and hence the degrees of freedom over perception are more limited.

The interplay between control and perception has recently received increasing attention. In [8], distance-dependent measurement models are leveraged to solve a combined estimation and control problem subject to probabilistic collision avoidance. In [5], the control actions are designed to explore an unknown environment while maximizing localization accuracy, but without accounting for the uncertainty evolution. Perception-aware control based on model predictive control (MPC) is presented in [9] to track a reference while maximizing the visibility of a point of interest. Similar concepts for planning the motion of an autonomous system, while concurrently maximizing the retention of obstacles or landmarks in the sensor range (e.g., the camera's field of view) have also been proposed [7, 14, 19–21, 24–26, 28]. Some approaches improve the estimation of the obstacles/targets by keeping them in the sensor field of view, including by optimizing the observability Gramians [21, 24]. In [15], a learning-based controller that combines perception and control has been presented that includes estimation of the uncertainty to assess unsafe conditions. Nonetheless, the majority of these works do not consider the impact of the varying environment uncertainty on the constraints, and hence on the controller, as the environment is explored. Furthermore, ensuring closed-loop stability in the presence of the internal feedback, where perception affects control and vice-versa, remains a non-trivial open problem.

In this paper, we consider a known nonlinear dynamical system that operates in an environment that is not known *a priori*. A given fixed estimator provides a stochastic estimate of the environment state based on information acquired from sensing, which depend on the system state and input. The system operation is not directly affected by process uncertainty, but rather by the uncertainty in the knowledge of environment, which affects the system operation through constraints relating the system state and the environment state. We design a perception-aware chance-constrained MPC (PAC-MPC) [1] that uses the environment estimation model to predict the evolution of the environment state and uncertainty, which is then used to enforce chance constraints [1, 11, 16, 18] between the system and environment. Thus, PAC-MPC accounts for the effects of the system operation on the perception quality, which may lead to a better closed-loop performance. For example, PAC-MPC may yield a trajectory that improves the sensing, which, in turn, reduces the uncertainty in the environment and thus reduces the tightening in the chance constraints, possibly resulting in less conservative future trajectories. We propose suitable designs for the cost function and terminal constraints, so that the PAC-MPC guarantees probabilistic recursive feasibility and stability of the system state to its target and of the environment estimate to its steady-state distribution. Although our method shares similarities with output-feedback MPC, see, e.g., [10], we consider a nonlinear system where, for the practical reasons discussed before, the control algorithm cannot modify the estimator, but only leverage its system actions-dependent performance. Furthermore, we leverage the problem structure, specifically the environment being dynamically decoupled from the system, to obtain stability results for system state and environment uncertainty, as opposed to convergence [10].

In what follows, Section 2 introduces the models for the system, the environment, its measurements, and its estimator. In Section 3, we describe the general design of PAC-MPC, and Section 4 provides general conditions for the closed-loop recursive feasibility and stability. In Section 5, we propose a constructive design for PAC-MPC for the linear setting that satisfies the conditions of Section 4. Section 6 presents the simulation results, followed by the conclusions in Section 7.

Notation: $\mathbb{R}$, $\mathbb{R}_{0+}$, $\mathbb{R}_{+}$, are the sets of real, nonnegative real, positive real numbers, respectively, and similar for integer numbers $\mathbb{Z}$. We denote interval of numbers with notations such as $\mathbb{Z}_{[a,b)} = \{z \in \mathbb{Z} : a \leq z < b\}$. Given vectors x y, the i-th component is $[x]_i$, the stacking is $(x, y) = [x'\ y']'$, $\|x\|$ is the 2-norm and $\|x\|^2_Q = x'Qx$. For a matrix X, $\|X\|_F$ is the Frobenious norm, where subscript may be dropped if clear from the context, and the trace is $\mathrm{tr}(X)$. $\mathbb{P}[A]$ is the probability of A. For a random vector x, $\mathbb{E}[x] = \mu^x$ is the expectation and Σ^x the covariance matrix. A normally distributed random vector is denoted by $x \sim \mathcal{N}(\mu^x, \Sigma^x)$. The moments may be grouped as $\mathcal{M}^x = (\mu^x, \Sigma^x)$. For a discrete-time signal $x \in \mathbb{R}^n$, x_k is the value at sampling instant k, $x_{j|k}$ is the predicted value at $k + j$ based on data at k, and $x_{0|k} = x_k$. A function $\alpha : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. In addition, if $\lim_{c\to\infty} \alpha(c) = \infty$, α is of class $\mathcal{K}_\infty$.

2 Modeling and Problem Definition

We consider a discrete-time system described by

$$x^s_{k+1} = f^s(x^s_k, u^s_k), \tag{1a}$$
$$y^s_k = q^s(x^s_k, u^s_k), \tag{1b}$$

where $x^s \in \mathbb{R}^{n_x}$ is the system state vector, $u^s \in \mathbb{R}^{n_u}$ is the system input vector, $y^s \in \mathbb{R}^{n_y}$ is the system performance output vector, $f^s : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is the system state update function and $q^s : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_y}$ is the system performance output function. System (1) and x^s are known at any time step, and are subject to

$$x^s \in \mathcal{X}, \quad u^s \in \mathcal{U}, \tag{2}$$

[1] This paper extends our works [2–4] by providing detailed proofs, a new method for controller design in the linear case that avoids previous restrictive assumptions, additional details on the algorithms, and extended simulation studies.

where $\mathcal{X}$ and $\mathcal{U}$ are the state and input admissible sets, respectively. We assume the following.

Assumption 1 *System* (1) *is controllable in* $\mathcal{X}$ *with input in* $\mathcal{U}$*, and observable with respect to* y^s. ■

Remark 1 *Throughout this paper, system* (1) *is assumed to be known, i.e., deterministic. System uncertainty may be accounted for by combining the proposed method with standard results on MPC for uncertain systems see [22, Ch.3] and references therein, e.g., by first tightening the constraints according to a tube MPC design, and then applying the methods described here.* ■

In addition to (2), the environment in which the system operates imposes additional constraints

$$h_l^s x^s + h_l^e x^e \leq h_l^b,\ l \in \mathbb{Z}_{[1,n_c,]}, \tag{3}$$

where $h_l^s \in \mathbb{R}^{n_x}$, $h_l^e \in \mathbb{R}^{m_x}$ are known vectors, and $x^e \in \mathbb{R}^{m_x}$ is the environment state vector describing variables imposing constraints on the system, such as positions of obstacles, velocities of other agents and angles of boundary markings. The environment evolution model is

$$x_{k+1}^e = f^e(x_k^e, \psi_k), \tag{4a}$$

where $\psi \in \mathbb{R}^{m_\psi}$ describes the exogenous inputs to (4a) and $f^e : \mathbb{R}^{m_x} \times \mathbb{R}^{m_\psi} \to \mathbb{R}^{m_x}$ is the environment state update function. Model (4) allows for representing stationary, e.g., boundary markings and fixed obstacles [3], and moving, e.g., cars and workers [4], elements of the environment by formulating constant dynamics or motion models, respectively. Thus, the environment state x^e contains all the information for modeling the position and/or motion of the environment elements as needed for prediction of the constraints. However, x^e is not directly known, hence its probability distribution is estimated based on perceived information,

$$y_k^e = q^e(x_k^e, \zeta_k, x_k^s, u_k^s), \tag{4b}$$

where $y^e \in \mathbb{R}^{m_y}$ is the environment measurement vector, $\zeta \in \mathbb{R}^{m_\zeta}$ is the measurement noise, and $q^e : \mathbb{R}^{m_x} \times \mathbb{R}^{m_\zeta} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{m_y}$ is the environment measurement function. We consider estimators that use the measurements from (4b) to provide the first two moments of a distribution of x^e, namely mean $\mu^e \in \mathbb{R}^{m_x}$ and covariance $\Sigma^e \in \mathbb{R}^{m_x \times m_x}$,

$$\mu_{k+1}^e = g_\mu \left(\mu_k^e, y_k^e, x_k^s, u_k^s\right), \tag{5a}$$
$$\Sigma_{k+1}^e = g_\Sigma \left(\Sigma_k^e, x_k^s, u_k^s\right), \tag{5b}$$

where $g_\mu : \mathbb{R}^{m_x} \times \mathbb{R}^{m_y} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{m_x}$ is the environment estimate mean update and $g_\Sigma : \mathbb{R}^{m_x \times m_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{m_x \times m_x}$ is the environment estimate covariance update. Although the environment evolution (4a) does not depend on (1), the estimate mean and covariance updates in (5) depend on x^s, u^s due to (4b). This enables capturing the dependency of the perception performance on the control decision, e.g., due to field of view and occlusions, sensor focus areas and levels, or if a sensor adjusts its gain based on focus or range. Often, the effects of such decisions are appreciable only in the uncertainty evolution, i.e., on the covariance update (5b), but (5a) allows for capturing also the effects in the mean. Since the environment exogenous input ψ in (4a) is not known, in (5) it is modeled as process noise, resulting in ψ, ζ being distributions described by the first two moments, μ^ψ, Σ^ψ, and μ^ζ, Σ^ζ, respectively. Such moments are not shown explicitly in (5) since here they are assumed to be constant for simplicity, though it is possible to extend to the time-varying case. In the remainder of the paper we will use the shorthand $g(\mathcal{M}^e, y^e, x^s, u^s)$ for the left hand side of (5).

Remark 2 *Estimator* (5) *is assumed fixed because in practical applications it is usually integrated in the sensing system and hence not a design choice for the controller. Here, we design a controller that leverages the dependency of* (5) *on* x^s, u^s*, to influence the estimate performance and achieve the control objective, and we provide suitable conditions on the estimator for this to succeed.* ■

While x^e is not directly known, (5) may include information on f^e, if available. The only requirement for (5) is to produce the first two moments μ^e, Σ^e of a valid distribution for x^e. Using only the first two moments allows for increased computational tractability, while additional moments can be included in a similar way.

Remark 3 *In the estimator* (5)*, the covariance update* (5b) *does not depend on the measurement* y^e*, e.g., as in Kalman filters. Instead,* (5a), (5b) *depend on the system states and inputs, which describe the dependence of the perception quality on the control actions.* ■

Now, we present the problem addressed in this paper.

Problem 1 *Consider system* (1) *subject to constraints* (2)*, environment* (4)*, environment estimator* (5) *and constraints* (3) *between system and environment. We want to design a control law that stabilizes* x^s *in an equilibrium where* $y^s = r$ *for a given output setpoint* $r \in \mathbb{R}^{n_y}$*, while satisfying* (2) *and* (3)*, the latter in a probabilistic sense due to the uncertainty in* (5). □

3 Perception-aware Chance Constrained MPC

We propose a perception-aware chance constrained MPC (PAC-MPC) for solving Problem 1. Since the environment state is not directly known, the environment prediction model is based on the estimator (5). However, (5) cannot be used for prediction because y^e is not known in advance. Thus, we predict mean and covariance of the estimate of x^e by the *environment predictor*

$$\hat{\mu}^e_{k+1} = \hat{g}_\mu\left(\hat{\mu}^e_k, \mu^y_k, x^s_k, u^s_k\right), \tag{6a}$$

$$\hat{\Sigma}^e_{k+1} = \hat{g}_\Sigma\left(\hat{\Sigma}^e_k, \Sigma^y_k, x^s_k, u^s_k\right), \tag{6b}$$

where $\hat{\mu}^e \in \mathbb{R}^{m_x}$, $\hat{\Sigma}^e \in \mathbb{R}^{m_x \times m_x}$ are the predicted mean and covariance of the estimate of x^e, $\hat{g}_\mu : \mathbb{R}^{m_x} \times \mathbb{R}^{m_y} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{m_x}$ and $\hat{g}_\Sigma : \mathbb{R}^{m_x \times m_x} \times \mathbb{R}^{m_y \times m_y} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{m_x \times m_x}$ are the environment estimate prediction mean and covariance update, respectively. In (6), μ^y is the predicted measurement mean according to the measurement prediction function $q^y : \mathbb{R}^{m_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{m_y}$

$$\mu^y_k = q^y(\mu^e_k, x^s_k, u^s_k), \tag{7}$$

and Σ^y is the covariance of the measurement prediction error, $\epsilon_y = y^e - \mu^y$. In what follows we will use the shorthand $\hat{g}(\mathcal{M}^{(e,y)}, x^s, u^s)$ to refer to the left hand side of (6). The following assumption is related to a "well-designed" estimator.

Assumption 2 *The mean estimator* (5a) *and mean predictor* (6a) *are asymptotically convergent, i.e.,* $\mu^e_k \to \bar{\mu}^e_\infty$*, and unbiased, i.e.,* $\mu^e_\infty = \mathbb{E}[x^e]$*, for every realization of the sequence* $\{(x^s_k, u^s_k)\}$. ■

Assumption 2 can be satisfied by proper estimator and predictor designs. In fact, since the system operation may affect the quality of the measurement (i.e., the covariance), but has minimal effects on the measurement itself (i.e., the mean), the mean estimate will usually not be greatly affected by the system state and input.

While constraints (2) involve only the state and input of (1), which are known, constraints (3) involve the environment state, for which a stochastic distribution is known from (5) and (6). Thus, (3) are enforced as individual chance constraints (ICCs)

$$\mathbb{P}\left[h^s_l x^s + h^e_l x^e \le h^b_l\right] \ge 1 - \varepsilon_l, \quad l \in \mathbb{Z}_{[1,n_c,]}, \tag{8}$$

where ε_l is the allowed probability of violation for the l^{th} constraint. The ICCs (8) can be formulated as tightened deterministic constraints [11, 18]

$$h^s_l x^s + [\gamma(\hat{\mathcal{M}}^e)]_l = h^s_l x^s + h^e_l \hat{\mu}^e + [\bar{\gamma}(\hat{\Sigma}^e)]_l \le h^b_l, \tag{9}$$

where γ is the impact of the environment on the constraints, and $\bar{\gamma}$ is the constraint tightening (backoff) parameter due to the environment estimate uncertainty.

The cost function of PAC-MPC encodes the objective of Problem 1. For a given reference $r_k \in \mathbb{R}^{n_y}$ for the system

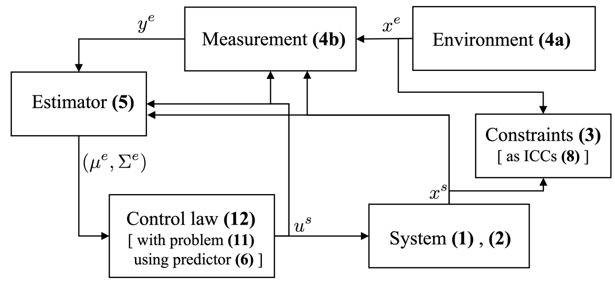

Fig. 1. Schematic of the relation between system, environment, estimator, and the control law of PAC-MPC.

performance output y^s, PAC-MPC aims at stabilizing x^s in an equilibrium such that $y^s_k = r_k$. The overall state consists of x^s, μ^e, Σ^e, where the mean of the environment estimate is guaranteed to converge independent of x^s, u^s by Assumption 2. Hence, PAC-MPC must stabilize the augmented state $\xi = (x^s, \Sigma^e)$. Accordingly, we define a cost function that includes a terminal cost $F : \mathbb{R}^{n_x} \times \mathbb{R}^{m_x \times m_x} \times \mathbb{R}^{n_y} \to \mathbb{R}_{0+}$ and stage cost $\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{m_x \times m_x} \times \mathbb{R}^{n_y} \to \mathbb{R}_{0+}$ such that

$$\begin{aligned} &V_N\left(x^s_k, U_k, \Sigma^e_k, r_k\right) = \\ &\quad F(x^s_{N|k}, \Sigma^e_{N|k}, r_k) + \sum_{j=0}^{N-1} \ell(x^s_{j|k}, u^s_{j|k}, \Sigma^e_{j|k}, r_k) = \\ &\quad F_c(x^s_{N|k}, r_k) + F_p(x^s_{N|k}, \Sigma^e_{N|k}, r_k) + \\ &\qquad \sum_{j=0}^{N-1} \ell_c(x^s_{j|k}, u^s_{j|k}, r_k) + \ell_p(x^s_{j|k}, u^s_{j|k}, \Sigma^e_{j|k}, r_k). \end{aligned} \tag{10}$$

In (10), $N \in \mathbb{Z}_+$ is the prediction horizon; $\ell_c : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_y} \to \mathbb{R}_{0+}$ and $F_c : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}_{0+}$ are the *control* stage and terminal costs, respectively, $\ell_p : \mathbb{R}^{n_u} \times \mathbb{R}^{m_x \times m_x} \times \mathbb{R}^{n_y} \to \mathbb{R}_{0+}$ and $F_p : \mathbb{R}^{m_x \times m_x} \times \mathbb{R}^{n_y} \to \mathbb{R}_{0+}$ are the *perception* stage and terminal cost, respectively, and $U_k = \left(u^s_{0|k}, \ldots, u^s_{N|k}\right)$.

Stabilizing the environment state estimate covariance, also prevents an uncontrolled increase of Σ^e, which would lead to large future tightening values in (9) and hence a possible reduced performance or even loss of feasibility. By combining the prediction models (1), (6), (7) and the constraints (2), (9), at each sampling time k, PAC-MPC solves the finite-horizon optimal control problem

$$
\begin{aligned}
&V_N^*(x_k^s, \mu_k^e, \Sigma_k^e, r_k) = \\
&\quad \min_{U_k} \; V_N\left(x_k^s, U_k, \Sigma_k^e, r_k\right) && \text{(11a)}\\
&\quad \text{s.t. } \; x_{j+1|k}^s = f^s(x_{j|k}^s, u_{j|k}^s) && \text{(11b)}\\
&\qquad \hat{\mathcal{M}}_{j+1|k}^e = \hat{g}(\hat{\mathcal{M}}_{j|k}^{(e,y)}, x_{j|k}^s, u_{j|k}^s) && \text{(11c)}\\
&\qquad \Sigma_{j+1|k}^e = g_\Sigma(\Sigma_{j|k}^e, x_{j|k}^s, u_{j|k}^s) && \text{(11d)}\\
&\qquad \mu_{j|k}^y = q^y(\mu_k^e, x_{j|k}^s, u_{j|k}^s) && \text{(11e)}\\
&\qquad (x_{j|k}^s, u_{j|k}^s) \in \mathcal{X} \times \mathcal{U} && \text{(11f)}\\
&\qquad h_\ell^s x_{j|k}^s + [\gamma(\hat{\mathcal{M}}_{j|k}^e)]_\ell \le h_i^b \; \ell \in \mathbb{Z}_{[1,n_c,]} && \text{(11g)}\\
&\qquad \left(x_{N|k}^s, r_k\right) \in \mathcal{Z}_f(\hat{\mathcal{M}}_{N|k}^e) && \text{(11h)}\\
&\qquad x_{0|k}^s = x_k^s, \; \mu_{0|k}^e = \mu_k^e, \; \Sigma_{0|k}^e = \hat{\Sigma}_{0|k}^e = \Sigma_k^e, && \text{(11i)}
\end{aligned}
$$

where $\mathcal{Z}_f \subseteq \mathbb{R}^{n_s} \times \mathbb{R}^{n_y}$ is the terminal set that depends on the first two moments of the environment estimate, and is used to ensure recursive feasibility, as described later. Denoting the solution of (11) by $U_k^* = (u_{0|k}^{s,*}, \dots, u_{N-1|k}^{s,*})$, the PAC-MPC law with block diagram shown in Fig. 1 is

$$u_k^s = \kappa_{\text{MPC}}\left(x_k^s, \mathcal{M}_k^e, r\right) = u_{0|k}^{s,*}. \tag{12}$$

Remark 4 *In* (11)*, there are two predictors for the environment estimate covariance:* (11d) *yields* Σ^e *for cost* (11a)*, while* (11c) *yields* $\hat{\Sigma}^e$ *for constraints* (11g)*. Since the future measurements* y_k^e *are unknown, the predictor* (6) *includes additional uncertainty by* Σ^y *to safely tighten the constraints* (11g) *Instead,* (5b) *does not require* y_k^e*, and hence* Σ^e *for cost* (11a) *can be predicted irrespective of the actual measurements. Both* $\Sigma_{0|k}^e$ *and* $\hat{\Sigma}_{0|k}^e$ *are initialized to* Σ_k^e*, in* (11i). ■

4 Recursive Feasibility and Stability Conditions

We now provide conditions for the design of the terminal set $\mathcal{Z}_f$ in (11h) and the terminal cost F in (11a) to achieve recursive feasibility and stability in probability.

4.1 Recursive Feasibility Conditions

For a stabilizing MPC [23], the terminal set must be positively invariant for (1), (5) in closed-loop with a terminal controller. We make the following assumptions.

Assumption 3 *Given any* y^e *from* (4b) *where* $\mathcal{M}^e = (\mu^e, \Sigma^e)$ *are the moments of the estimate of* x^e*,* $\gamma(\hat{g}(\mathcal{M}^{(e,y)}, x^s, u^s)) \ge \gamma(g(\mathcal{M}^e, y^e, x^s, u^s))$ *for all* x^s*,* u^s*.* ■

Assumption 4 *Given any two* $\mathcal{M}_1^e = (\mu_1^e, \Sigma_1^e)$*,* $\mathcal{M}_2^e = (\mu_2^e, \Sigma_2^e)$ *such that* $\gamma(\mathcal{M}_1^e) \ge \gamma(\mathcal{M}_2^e)$*,* $\gamma(\hat{g}(\mathcal{M}_1^{(e,y)}, x^s, u^s)) \ge \gamma(\hat{g}(\mathcal{M}_2^{(e,y)}, x^s, u^s))$ *for all* x^s*,* u^s*.* ■

Assumption 3 ensures that the predictor does not underestimate the constraint tightening with respect to the environment mean and covariance estimate, and the inequality can be satisfied by the choice of Σ^y in accordance to the measurement gain in the estimator (5). Assumption 4 requires that for any two sets of moments, the one requiring a larger constraint tightening will result in a prediction that also imposes a larger constraint tightening than the prediction of the other one. Thus, if the uncertainty "mass" accounted for in the chance constraint is larger in one case when compared to another case, it will remain larger after prediction is operated on both. This amounts to (6) applying non-abrupt updates to the estimate moments, and is satisfied when the estimator gain is not excessively large.

Remark 5 *While ideally* Σ^y *is chosen equal to the covariance of the measurement prediction error, for the results developed in the remainder of this paper it suffices to choose* Σ^y *such that Assumption 3 holds. A method that leverages scenario-based optimization in combination with PAC-MPC to reduce the conservativeness of* Σ^y *at the price of an increased computational burden in the optimal control problem was presented in [4].* ■

Theorem 1 *Let Assumptions 3, 4 hold. Let there exist a control law* $\kappa_f : \mathbb{R}^{n_x} \times \mathbb{R}^{m_x} \times \mathbb{R}^{m_x \times m_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_u}$ *and a set* $\mathcal{Z}_f(\mathcal{M}^e)$ *such that if* $(x^s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$*,*

$$
\begin{aligned}
(i) \quad & x^s \in \mathcal{X}, \; \kappa_f(x^s, \mathcal{M}^e, r) \in \mathcal{U}, \\
& \mathbb{P}\left[h_i^s x^s + h_i^e x^e \le h_i^b\right] \ge 1 - \varepsilon_i, \; i \in \mathbb{Z}_{[1,n_c,]}, \\
(ii) \quad & (f^s(x^s, \kappa_f(x^s, \mathcal{M}^e, r)), \; r) \\
& \qquad \in \mathcal{Z}_f(\hat{g}(\mathcal{M}^{(e,y)}, x^s, \kappa_f(x^s, \mathcal{M}^e, r))).
\end{aligned}
$$

Then, if (11) *is feasible at time* k *for* (1)*,* (5) *in closed-loop with* (12) *and* $r_{k+1} = r_k$*,* (11) *is feasible at time* $k+1$ *with probability at least* $\prod_{i=1}^{n_c} \varepsilon_i$*.*

Proof 1 *By* (i)*, if* $(x^s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$*, it also satisfies* (2) *and* (8)*. Let the solution at time* k *be* $U_k^* = (u_{0|k}^{s,*} \dots u_{N-1|k}^{s,*})$*, yielding the state trajectory* $X_k^* = (x_{0|k}^{s,*} \dots x_{N|k}^{s,*})$ *and* $\Gamma_k^* = (\gamma_{0|k}^* \dots \gamma_{N|k}^*)$*. By the ICCs* (11g)*, the probability that the initial state at step* $k+1$ *satisfies the constraints is* $\prod_{i=1}^{n_c} \varepsilon_i$*. Using the solution at time* k *and the terminal controller* $\kappa_f(\cdot)$*, we construct a candidate solution at* $k+1$*, i.e.,* $\tilde{U}_{k+1} = (u_{1|k}^{s,*} \dots u_{N-1|k}^{s,*} \quad \kappa_f(x_{N|k}^s, \hat{\mathcal{M}}_{N|k}^e, r))$*, which yields* $\tilde{X}_{k+1} = (x_{1|k}^{s,*} \dots x_{N|k}^{s,*} \quad f^s(x_{N|k}^{s,*}, \kappa_f(x_{N|k}^{s,*}, \hat{\mathcal{M}}_{N|k}^e, r)))$*. For* $\tilde{\Gamma}_{k+1}$*, using* $\tilde{X}_{k+1}$*,* $\tilde{U}_{k+1}$*, from* (6) *we obtain*

$\tilde{\gamma}_{j+1|k+1} = \gamma(\hat{g}(\hat{\mathcal{M}}^{(e,y)}_{j|k+1}, \tilde{x}^s_{j|k+1}, \tilde{u}^s_{j|k+1}))$. *By Assumption 3,* $\gamma(\mathcal{M}^e_{k+1}) \leq \gamma(\hat{\mathcal{M}}^e_{1|k})$, *and by combining Assumptions 3, 4,* $\gamma(\hat{\mathcal{M}}^e_{j|k+1}) \leq \gamma(\hat{\mathcal{M}}^e_{j+1|k})$ *for all* $j \in \mathbb{Z}_{[1,\ldots N-1]}$. *Thus, the constraint tightening does not increase, and hence all the constraints at steps* $j \in \mathbb{Z}_{[0,\ldots N-1]}$ *are satisfied. Since by (ii)* κ_f *makes the terminal set invariant for* (1), (6), *the terminal constraint* (11h) *is also satisfied. Thus, if* x_{k+1} *satisfies the constraints, which has a probability at least* $\prod_{i=1}^{n_c} \varepsilon_i$, *a feasible solution of* (11) *exists.* ■

In Theorem 1, (i) requires $\mathcal{Z}_f$ to be contained in the set where κ_f satisfies (2), (8) and (ii) requires it be invariant for (1) in closed-loop with κ_f and (6), where Assumptions 3, 4 ensure invariance with the constraint tightening due to environment prediction. Although Theorem 1 is general, a design of the terminal controller and terminal set to satisfy such assumptions may be challenging, especially since the conditions concurrently involve the design of the predictor and the terminal controller. If (6) satisfies additional properties, a decoupled design of the terminal controller κ_f and the predictor $\hat{g}$ achieves similar properties.

Corollary 1 *Consider* $\kappa_f(x^s, \mathcal{M}^e, r) = \kappa_f(x^s, r)$, $\mathcal{Z}_f(\mathcal{M}^e) = \bar{\mathcal{Z}}_f(\gamma(\mathcal{M}^e))$ *such that if* $\gamma(\mathcal{M}^e_1) \leq \gamma(\mathcal{M}^e_2)$, *then* $\bar{\mathcal{Z}}_f(\gamma(\mathcal{M}^e_1)) \supseteq \bar{\mathcal{Z}}_f(\gamma(\mathcal{M}^e_2))$. *Let Assumptions 3, 4 hold, and let* (6) *be such that* $\gamma(\hat{g}(\mathcal{M}^{(e,y)}, x^s, \kappa_f(x^s, r))) \leq \gamma(\mathcal{M}^e)$ *for all* $(x_s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$. *Let* $\kappa_f(x^s, r)$, $\mathcal{Z}_f(\mathcal{M}^e)$ *be such that, if* $(x^s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$,

$$\begin{aligned} (i) \quad & x^s \in \mathcal{X},\ \kappa_f(x^s, r) \in \mathcal{U}, \\ & \mathbb{P}\left[h^s_i x^s + h^e_i x^e \leq h^b_i\right] \geq 1 - \varepsilon_i,\ i \in \mathbb{Z}_{[1,n_c]}, \\ (ii) \quad & (f^s(x^s, \kappa_f(x^s, r)),\ r) \in \mathcal{Z}_f(\mathcal{M}^e). \end{aligned}$$

Then, if (11) *is feasible at time* k *and* $r_{k+1} = r_k$ *for* (1), (5) *in closed-loop with* (12), (11) *is feasible at time* $k+1$ *with probability greater or equal to* $\prod_{i=1}^{n_c} \varepsilon_i$.

Proof 2 *As for Theorem 1, we start by using* (i) *to note that, if* $(x^s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$, *it satisfies constraints* (2), (8). *Following the reasoning of Theorem* (1), *since now* $\kappa_f(\cdot)$ *does not depend on* $\mathcal{M}^e$, *the candidate solution at* $k+1$ *is* $\tilde{U}_{k+1} = (u^{s,*}_{1|k} \ldots u^{s,*}_{N-1|k}\ \kappa_f(x^s_{N|k}, r))$, *which yields* $\tilde{X}_{k+1} = (x^{s,*}_{1|k} \ldots x^{s,*}_{N|k}\ f^s(x^{s,*}_{N|k}, \kappa_f(x^s_{N|k}, r)))$. *Again, we obtain* $\tilde{\Gamma}_{k+1}$ *from* $\tilde{X}_{k+1}$, $\tilde{U}_{k+1}$, (6) *as* $\tilde{\gamma}_{j+1|k+1} = \gamma(\hat{g}(\hat{\mathcal{M}}^{(e,y)}_{j|k+1}, \tilde{x}^s_{j|k+1}, \tilde{u}^s_{j|k+1}))$, *and* $\gamma(\hat{\mathcal{M}}^e_{j|k+1}) \leq \gamma(\hat{\mathcal{M}}^e_{j+1|k})$ *for all* $j \in \mathbb{Z}_{[1,\ldots N-1]}$ *by combining Assumptions 3 and 4. Hence, all the constraints are satisfied up to the terminal constraint. For the latter, if* $(x^{s,*}_{N|k}, r) \in \mathcal{Z}_f(\hat{\mathcal{M}}^e_{N|k})$, *then* $(f^s(x^{s,*}_{N|k}, \kappa_f(x^{s,*}_{N|k}, r)),\ r) \in \mathcal{Z}_f(\hat{\mathcal{M}}^e_{N|k}) \subseteq \mathcal{Z}_f(\hat{\mathcal{M}}^e_{N|k+1})$. *Thus there exists a feasible solution for* (11) *as long as* x^s_{k+1} *satisfies the constraints in the initial step, which has probability at least* $\prod_{i=1}^{n_c} \varepsilon_i$. ■

In Corollary 1, the terminal control law κ_f does not directly depend on the environment information $\mathcal{M}^e$. The terminal set $\mathcal{Z}_f$ depends on $\mathcal{M}^e$ only through the constraint tightening γ, and $\mathcal{Z}_f$ does not shrink as γ decreases.

Remark 6 *In the proofs of Theorem 1 and Corollary 1, Assumptions 3, 4 ensure that the previously predicted PAC-MPC trajectory remains feasible when the one-step ahead environment prediction is substituted with the updated environment estimate, i.e., the tightening of* (3) *due to ICC* (8) *does not expand. If Assumptions 3, 4 do not hold, besides a standard constraint softening, one may modify the tightening* $\gamma(\hat{\mathcal{M}}^e_{j|k})$ *to be the slacks of* (3) *for the previously predicted PAC-MPC trajectory, when the environment prediction is initialized by the updated estimate from* (5). *This ensures that the previous PAC-MPC solution is still feasible for* (11), *although the probability of satisfying* (3) *may be lower than the one in* (8). *As in [10], this may be done if the solution to* (11) *with tightening from* (8) *results in infeasibility.* ■

4.2 Stability Conditions

Next, we investigate conditions under which the control law (12) stabilizes (1) with the environment state estimate provided by (5). In the rest of this section, for simplicity of notation, $r_k = 0$ for all $k \in \mathbb{Z}_{0+}$, and hence omitted. The full state of (1), (5) is $\varphi = (x^s, \mu^e, \Sigma^e)$, where μ^e, Σ^e affect only the ICCs (9). Due to Assumption 2, μ^e asymptotically converges to $\mathbb{E}[x^e]$, and we only need conditions that stabilize $\xi = (x^s, \Sigma^e)$ to an equilibrium $\xi^r = (x^r, \Sigma^r)$, being the equilibrium state for (1) and the equilibrium covariance (i.e., uncertainty) for (5).

Proposition 1 *The function*

$$\|\xi\| = \|x^s\| + \|\Sigma^e\|_F \tag{13}$$

is a norm for $\xi = (x^s, \Sigma^e)$.

Proof 3 *The function is nonnegative, since it sums the norms of* x^s *and* $\|\Sigma^e\|_F$, *and it is* 0 *only if both are* 0, *i.e., if* $\xi = (0, 0)$. *Given* $\xi_1 = (x^s_1, \Sigma^e_1)$, $\xi_2 = (x^s_2, \Sigma^e_2)$, $\|\xi_1 + \xi_2\| = \|x_1 + x_2\| + \|\Sigma^e_1 + \Sigma^e_2\| \leq \|x_1\| + \|\Sigma^e_1\| + \|x_2\| + \|\Sigma^e_2\| = \|\xi_1\| + \|\xi_2\|$. ■

We denote the dynamics of (1), (5) by $\varphi_{k+1} = \Phi(\varphi_k, u^s_k, y^e_k)$, by ς the function that selects ξ from φ, i.e., $\varsigma(\varphi) = \varsigma((x^s, \mu^e, \Sigma^e)) = (x^s, \Sigma^e) = \xi$, $\Phi^\xi = \varsigma \circ \Phi$, and we write (10) as

$$V_N(\xi, U) = F(\xi_{N|k}) + \sum_{j=0}^{N-1} \ell(\xi_{j|k}, u^s_{j|k}), \qquad (14)$$

where $F(\xi) = F_c(x^s) + F_p(\Sigma^e)$ and $\ell(\xi, u^s) = \ell_c(x^s, u^s) + \ell_p(\Sigma^e)$. The value function of (11) is $V_N^*(\varphi) = V_N^*(x^s, \mu^e, \Sigma^e)$.

Assumption 5 *The control stage cost is such that $\ell_c(x^s, 0) \leq \ell_c(x^s, u^s)$ for all $u^s \in \mathcal{U}$ and there exist functions $\alpha_l^c, \alpha_l^p, \alpha_u^c, \alpha_u^p \in \mathcal{K}_\infty$ such that $\ell_c(x^s, 0) \geq \alpha_l^c(\|x^s\|)$, $F_c(x^s) \leq \alpha_u^c(\|x^s\|)$ and $\ell_p(\Sigma^e) \geq \alpha_l^p(\|\Sigma^e\|)$, $F_p(\Sigma^e) \leq \alpha_u^p(\|\Sigma^e\|)$.* ■

Assumption 6 *The control law $u = \kappa_f(\varphi)$ is such that for all $x^s \in \mathcal{Z}_f(\mathcal{M}^e)$, $F(\xi) \geq \ell(\xi, \kappa_f(\varphi)) + F(\Phi^\xi(\varphi, \kappa_f(\varphi), y^e))$ and*

$$f^s(x^s, \kappa_f(\varphi)) \in \mathcal{Z}_f(\hat{g}(\mathcal{M}^e, \mathcal{M}^y, x^s, \kappa_f(\varphi))). \qquad ■$$

Assumption 5 is standard for MPC cost functions (11a), see [22], and Assumption 6 is related to the existence of local control Lyapunov function, and will be satisfied by the designs proposed later.

We first prove that under Assumptions 5, 6 $V_N^*(\varphi)$, is a Lyapunov function of ξ for (1) in closed-loop with (12).

Lemma 1 *Let Assumptions 5, 6 hold, then there exist functions $\alpha_l, \alpha_u, \alpha_\Delta \in \mathcal{K}_\infty$ such that*

$$\alpha_l(\|\xi\|) \leq V_N^*(\varphi) \leq \alpha_u(\|\xi\|), \qquad (15a)$$
$$V_N^*(\Phi(\varphi, \kappa_{\text{MPC}}(\varphi), y^e)) - V_N^*(\varphi) \leq -\alpha_\Delta(\|\xi\|), \qquad (15b)$$

whenever (11) is feasible for $(x_k^s, \mu_k^e, \Sigma_k^e) = \varphi$ and for $\Phi(\varphi, \kappa_{\text{MPC}}(\varphi), y^e)$.

Proof 4 *By construction $V_N^*(\varphi) \geq \ell(\xi, 0)$. Under the assumptions, we can choose*

$$\alpha_l(\|\xi\|) = \min\{(\alpha_l^c(1/2\|\xi\|), \alpha_l^p(1/2\|\xi\|))\}$$

since by (13) $\|\Sigma^e\| \geq 1/2\ \|\xi\|$ for $\|\Sigma^e\| \geq \|x^s\|$, and the opposite holds when $\|x^s\| \geq \|\Sigma^e\|$. Thus,

$$\alpha_l(\|\xi\|) \leq \alpha_l^c(1/2\|\xi\|) \leq \alpha_l^p(\|\Sigma^e\|) \leq \ell_p(\Sigma^e) \leq \ell(\xi, 0) \leq V_N^*(\varphi).$$

For the upper bound, [23] guarantees that there exists $c > 0$ such that $cF(\xi) \geq V_N^*(\varphi)$. Then,

$$V_N^*(\varphi) \leq cF_c(x^s) + cF_p(\Sigma^e) \leq c\alpha_u^c(\|x^s\|) + c\alpha_u^p(\|\Sigma^e\|) \leq c\alpha_u^c(\|\xi\|) + c\alpha_u^p(\|\xi\|) = \alpha_u(\|\xi\|).$$

By [23], if $F(\xi) \geq \ell(\xi, \kappa_f(\varphi)) + F(\Phi^\xi(\varphi, \kappa_f(\varphi), y^e))$, then

$$V_N^*(\Phi(\varphi, \kappa_{\text{MPC}}(\varphi), y^e)) - V_N^*(\varphi) \leq -\ell(\xi, \kappa_{\text{MPC}}(\varphi)) \leq -\ell(\xi, 0) \leq -\alpha_l(\|\xi\|) = -\alpha_\Delta(\|\xi\|).$$

■

Theorem 2 *Let Assumption 5 and the conditions of Theorem 1 or Corollary 1 hold. If for all $x_s \in \mathcal{Z}_f(\mathcal{M}^e)$*

$$F_c(f^s(x^s, \kappa_f(\varphi)) - F_c(x^s) + \ell_c(x^s, \kappa_f(\varphi)) \leq -M(x^s) \qquad (16a)$$
$$F_p(g_\Sigma(\Sigma^e, x^s, \kappa_f(\varphi))) - F_p(\Sigma^e) + \ell_p(\xi, \kappa_f(\varphi)) \leq M(x^s) \qquad (16b)$$

are satisfied, where M is a nonnegative function, at every step the closed-loop (1), (5), (12) has probability at least $\prod_{i=1}^{n_c} \varepsilon_i$ to evolve according to the Lyapunov function (15).

Proof 5 *Theorem 1 and Corollary 1 ensure that (11) is recursively feasible with probability $\prod_{i=1}^{n_c} \varepsilon_i$ at every step and that κ_f makes the terminal set invariant, i.e., $f^s(x^s, \kappa_f(\varphi)) \in \mathcal{Z}_f(\hat{g}(\mathcal{M}^{(e,y)}, x^s, \kappa_f(\varphi)))$. If (16) holds, $F(\xi) \geq \ell(\xi, \kappa_f(\varphi)) + F(\Phi^\xi(\varphi, \kappa_f(\varphi), y^e))$. Thus, the assumptions of Lemma 1 are all satisfied. Hence, with probability equal to that of satisfying the chance constraints, $\prod_{i=1}^{n_c} \varepsilon_i$, the closed-loop dynamics evolve according to the Lyapunov function (15) for $\xi = \varsigma(\varphi)$.* ■

Remark 7 *Theorem 2 proves the stability of $\xi = (x^s, \Sigma^e) = \varsigma(\varphi)$, since we consider that the system operation affects the measurement quality, i.e., the covariance, and not its value, i.e., the mean. Proving the stability of φ also requires guaranteeing the stability of the mean estimate μ^e, by modifying Assumption 2.* ■

5 Constructive Design for Linear Dynamics

The conditions in Section 4 for recursively feasibility and closed-loop stability are established for a general nonlinear system and, hence, it is hard to derive a general constructive procedure to satisfy them. Next, we derive a constructive design procedure for achieving the recursive feasibility and stability properties of Section 4 for the case in which (1), (2), (4), (5), (6) are linear. Let (1) be the linear system

$$x_{k+1}^s = A^s x_k^s + B^s u_k^s, \qquad (17a)$$
$$y_k^s = E^s x_k^s, \qquad (17b)$$

and the constraints in (2) be polyhedral

$$H^x x^s \leq K^x,\ H^u u^s \leq K^u. \qquad (18)$$

For (5), we consider a linear estimator based on the open-loop environment model

$$x^e_{k+1} = A^e x^e_k + B^e \psi_k, \tag{19a}$$
$$y^e_k = C^e(x^s_k, u^s_k)x^e_k + D^e(x^s_k, u^s_k)\zeta_k, \tag{19b}$$

where $\psi_k \sim \mathcal{N}(\mu^\psi, \Sigma^\psi)$ and $\zeta_k \sim \mathcal{N}(\mu^\zeta, \Sigma^\zeta)$. The environment estimate mean and covariance evolve as

$$\mu^e_{k+1} = \Lambda_k \mu^e_k + B^e \mu^\psi - L_k y^e_k, \tag{20a}$$
$$\Sigma^e_{k+1} = \Lambda_k \Sigma^e_k \Lambda'_k + Q + R_k, \tag{20b}$$

where $C_k = C^e(x^s_k, u^s_k)$, $D^e_k = D^e(x^s_k, u^s_k)$, $L_k = L(x^s_k, u^s_k)$, $\Lambda_k = \Lambda(x^s_k, u^s_k) = A^e + L_k C_k$, $Q = B^e \Sigma^\psi B^{e\prime}$, and $R(x^s_k, u^s_k) = L_k D^e_k \Sigma^\zeta (L_k D^e_k)'$. As in (4), the measurement (19b) depends on the system states and inputs to describe the variable perception quality. Based on (20), the predictor (6) takes the form

$$\hat{\mu}^e_{k+1} = \Lambda_k \hat{\mu}^e_k + B^e \mu^\psi - L_k \mu^y_k, \tag{21a}$$
$$\hat{\Sigma}^e_{k+1} = \Lambda_k \hat{\Sigma}^e_k \Lambda'_k + Q + \hat{R}_k, \tag{21b}$$

where $\hat{R}_k = \hat{R}(x^s_k, u^s_k) = L_k(D^e_k \Sigma^\zeta D^{e\prime}_k + \Sigma^y_k)L'_k$ to account for the measurement prediction error.

We consider a constant reference r for y^s. Under Assumption 1, given r, by setting $[r^u]_i$ to nominal values for the inputs $[u]_i$ that only affect (19b), i.e., perception, we obtain unique constant setpoints r^x and r^u for x^s and u^s, respectively. Next, we design the terminal constraint (11h) and cost (11a) such that (17), (18), (19), (8) in closed-loop with (12), (20) satisfy the properties of Section 4.

5.1 Terminal Set Design

Due to linearity of (17), (18), we consider terminal controllers of the form

$$u = \kappa_f(x^s, r) = K_f x^s + G_f r, \tag{22}$$

where K_f is a stabilizing gain for (17) and G_f provides unitary dc-gain from r to y when (17) is in closed-loop with (22). The challenge in constructing the terminal set using [13] is that (9) depends nonlinearly on Σ^e, which is not constant and cannot be predicted open-loop since it is affected by the control actions. Let the admissible references be constrained by $H^r r \leq K^r$, we construct the maximum constrained admissible set (MCAS) [13] for (17) in closed-loop with (22),

$$\mathcal{O}_\infty = \{(x^s_0, r_0, \eta) : \ H^x x^s_k \leq K^x, \ H^u \kappa_f(x^s, r) \leq K^u, \\ H^r r_k \leq K^r, h^s_i x^s_k + [\eta]_i \leq h^b_i, i \in \mathbb{Z}_{[1,n_c,]}, \ \forall k \in \mathbb{Z}_{0+}\}, \tag{23}$$

with the additional dynamics $r_{k+1} = r_k = r$ and $\eta_{k+1} = \eta_k$. Set (23) is the MCAS of a lifted system, where η describes the tightening margin on the ICCs. We define $\mathcal{O}_\infty(\eta) = \{(x^s, r) : (x^s, r, \eta) \in \mathcal{O}_\infty\}$.

Corollary 2 *Let* (1), (2) *be* (17), (18), *respectively. Let* $\mathcal{Z}_f(\mathcal{M}^e) = \mathcal{O}_\infty(\gamma(\mathcal{M}^e))$, (6) *be such that for all* $(x_s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$, $\gamma(\hat{g}(\mathcal{M}^{(e,y)}, x^s, \kappa_f(x^s, r))) \leq \gamma(\mathcal{M}^e)$. *If Assumptions 3, 4 hold and* (11) *is feasible at time* k *and* $r_{k+1} = r_k$, (11) *is feasible at time* $k+1$ *for* (17), (20), (12) *with probability greater or equal to* $\prod_{i=1}^{n_c} \varepsilon_i$.

Proof 6 *We prove the result by showing that, under the stated assumptions, the assumptions of Corollary 1 hold, which then provide the result. The choice of* $\mathcal{Z}_f(\mathcal{M}^e)$ *satisfies:* (*i*) *in Corollary 1 by the definition of* (23) *when* $\eta = \gamma(\mathcal{M}^e)$*; and* (*ii*) *in Corollary 1 since the MCAS is positively invariant [13] for* (17) *in closed-loop with* (22). *In addition,* $\mathcal{O}_\infty(\eta_1) \supseteq \mathcal{O}_\infty(\eta_2)$ *when* $\eta_2 \geq \eta_1$ *due to the monotonicity of* $\mathcal{O}_\infty$ *with respect to the admissible region. Due to the remaining assumptions, all the assumptions of Corollary 1 are satisfied, hence proving the statement.* ■

5.2 Stabilizing Terminal Cost Design

To provide constructive stability conditions, we consider the case where $\Lambda(x^s, u^s) = \Lambda = A^e + LC^e$, i.e., the estimation error update matrix does not depend on the system state and input. The dependency on state and input can still be present in $R(x^s, u^s)$, as well as in L and C^e, if it cancels out in the product. For the linear case, in (10) we consider the control stage and terminal costs

$$\ell_c(x^s, u^s, r) = ||x^s - r^x||^2_{Q_c} + ||u^s - r^u||^2_{R_c}, \tag{24a}$$
$$F_c(x^s, r) = ||x^s - r^x||^2_{P_c}, \tag{24b}$$

where $Q_c, R_c, P_c > 0$ are weight matrices. For the perception cost, we construct the steady-state environment covariance by

$$\Sigma^r = \Lambda \Sigma^r \Lambda' + Q + R(r^x, r^u).$$

For $\bar{\Sigma}^e = \Sigma^e - \Sigma^r$, $\bar{R}(x^s, u^s) = R(x^s, u^s) - R(r^x, r^u)$, we obtain the error of the covariance matrix with respect to the steady-state as $\bar{\Sigma}^e_{k+1} = \Lambda \bar{\Sigma}^e_k \Lambda' + \bar{R}(x^s_k, u^s_k)$. The perception stage and terminal costs in (10) are chosen as

$$\ell_p(\Sigma^e_k) = S_c \|\bar{\Sigma}^e_k\|^2_F, \tag{25a}$$
$$F_p(\Sigma^e_k) = W_c \sum_{h=0}^{N_p-1} \rho^h \|\Lambda^h \bar{\Sigma}^e_k \Lambda^{h\prime}\|^2_F, \tag{25b}$$

where $N_p \in \mathbb{Z}_+$, $\rho \in \mathbb{R}_{[1,\infty)}$ are design parameters. The terminal perception cost (25a) is the sum of the environment covariance matrix errors, over horizon N_p, from the terminal state at the end of the prediction horizon for the closed-loop (17), (22). Thus, (25a) is the finite-horizon approximation of the perception cost-to-go. We allow $N_p \geq 1$ because if $N_p = 1$ as in [2], the cost decrease condition requires the Frobenius norm of the covariance error to contract in a single step, which in turn requires $\|\Lambda\|_F < 1$ that is not always possible to achieve. Instead, with a convergent estimator (Assumption 2), there always exists $N_p \in \mathbb{Z}_+$ such that $\|\Lambda^{N_p}\|_F < 1$.

In the general case, N_p may need to be computed by simulations. However, when $\Lambda(x^s, u^s) = A^s + LC^s$, it is straightforward to find a value for N_p that satisfies the condition.

Corollary 3 *Consider* (17), (18), *and* (22) *and the environment* (19). *Let the environment estimator be* (20) *where* $\Lambda(x^s, u^s) = \Lambda$, $\mathcal{Z}_f(\mathcal{M}^e) = \mathcal{O}_\infty(\gamma(\mathcal{M}^e))$ *and the assumptions of Corollary 2 hold. If there exist* $W_c, S_c \in \mathbb{R}_+$, $\varrho \in \mathbb{R}_+$, $M_c \geq 0$, $N_p \in \mathbb{Z}_+$, *such that for all* $(x_s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$,

$$\rho \geq (1+\varrho), \tag{26a}$$

$$\rho^{N_p}\|\Lambda^{N_p}\|_F^2 \leq 1 - S_c/W_c, \tag{26b}$$

$$\frac{x^{s\prime} M_c x^s}{(1+\varrho^{-1})W_c} \geq \sum_{i=0}^{N_p-1} \rho^h \|\Lambda^h \bar{R}(x^s, \kappa_f(x^s, r))\Lambda^{h\prime}\|_F^2, \tag{26c}$$

$$\begin{aligned} &P_c - (A^s + B^s K_f)' P_c (A^s + B^s K_f) \geq \\ &\qquad K_f' R_c K_f + Q_c + M_c, \end{aligned} \tag{26d}$$

then with (12), *the evolution of* $\xi = (x^s, \Sigma^e)$ *satisfies* (15) *at each step with probability at least* $\prod_{i=1}^{n_c} \varepsilon_i$.

Proof 7 *We prove that the conditions of Theorem 2 hold. Due to* κ_f *satisfying Corollary 2, the terminal controller has the properties in Corollary 1. Next, we show that* (16) *holds. As in Theorem 2, we consider* $r_k = r_{k+1} = 0$ *for simplicity and omit it. Choosing* $M(x^s) = x^{s\prime} M_c x^s$, (16a) *amounts to* (26d). *For* (16b), *let* $\bar{R}_f = \bar{R}(x^s, \kappa(x^s))$ *and* $\Sigma_+^e = \Lambda\Sigma^e\Lambda' + Q + R_f(x^s)$, *then*

$$\begin{aligned} F_p(\Sigma_+^e) &= W_c \sum_{h=0}^{N_p-1} \rho^h \|\Lambda^h(\bar{R}_f(x^s) + \Lambda\bar{\Sigma}^e\Lambda')\Lambda^{h\prime}\|_F^2 \\ &= W_c \sum_{h=1}^{N_p} \rho^{h-1} \|\Lambda^h\bar{\Sigma}^e\Lambda^{h\prime} + \Lambda^{h-1}\bar{R}_f(x^s)\Lambda^{h-1\prime}\|_F^2 \\ &\leq W_c \sum_{h=1}^{N_p} \rho^{h-1}((1+\varrho)\|\Lambda^h\bar{\Sigma}^e\Lambda^{h\prime}\|_F^2 \\ &\qquad + (1+\varrho^{-1})\|\Lambda^{h-1}\bar{R}_f(x^s)\Lambda^{h-1\prime}\|_F^2) \\ &\leq W_c \sum_{h=1}^{N_p} \rho^h \|\Lambda^h\bar{\Sigma}^e\Lambda^{h\prime}\|_F^2 + M(x^s), \end{aligned}$$

where we used Young's inequality (first upper bounding) that holds for every $\varrho \in \mathbb{R}^+$, *and* (26d) *(second upper bounding). Thus,*

$$\begin{aligned} &F_p(\Sigma_+^e) + \ell_p(\Sigma^e) - F_p(\Sigma^e) - M(x^s) \\ &\quad \leq S_c\|\bar{\Sigma}^e\|_F^2 - W_c\|\bar{\Sigma}^e\|_F^2 + \rho^{N_p}W_c\|\Lambda^{N_p}\bar{\Sigma}^e\Lambda^{N_p\prime}\|_F^2 \\ &\quad \leq W_c\|\bar{\Sigma}^e\|_F^2(\rho^{N_p}\|\Lambda^{N_p}\|_F^2 + S_c/W_c - 1). \end{aligned}$$

Since $\rho^{N_p}\|\Lambda^{N_p}\|_F^2 \leq 1 - S_c/W_c$ *by* (26b), (16b) *holds and all the conditions of Theorem 2 hold.* ■

Conditions (26) of Corollary 3 require choosing N_p such that $\|\Lambda^{N_p}\|_F^2 < c/\rho_p^N$, $c < 1$. By Assumption 2, $\lim_{h\to\infty}\|\Lambda^h\|_F^2 = 0$, and hence it is always possible to satisfy it. The coefficient ρ is related to ϱ in Young's inequality from (26a). We could set $\rho = 1$ if the (non-squared) Frobenius norm is used in (25), which would result in a more challenging optimization problem. Condition (26c) bounds the increase in the perception cost due to the difference between $R(x^s, \kappa_f(x^s, r))$ and the steady-state $R(r^x, r^u)$. Such bound is accounted for in (26d) to ensure that any increase in perception cost is compensated by a larger decrease of the control cost. M_c and K_f can be determined iteratively, possibly by simulation and linear regression, since they only depend on an initial state x^s, where $(x^s, r) \in \mathcal{Z}_f(\mathcal{M}^e)$.

Remark 8 *Extending the design to a general* $\Lambda(x^s, u^s)$ *requires considering the* h*-steps state transition matrices* $\prod_{\ell=0}^{h} \Lambda(x^s_{(\ell)}, \kappa_f(x^s_{(\ell)}, r))$, *where* $x^s_{(\ell)}$ *is the* ℓ*-steps ahead prediction of* x^s *based on* (17), (22) *for the time-varying system, as opposed to* Λ^h, *and defining a reference trajectory for the covariance matrix that converges to the setpoint, so that one can express the error of the covariance matrix as* $\bar{\Sigma}_{k+1}^e = \Lambda(x^s, \kappa_f(x^s))\bar{\Sigma}_k^e\Lambda(x^s, \kappa_f(x^s))' + \bar{R}(x_k^s, u_k^s)$. ■

Remark 9 *For perception cost* $\ell_p(\xi, u) = S_c\,(tr(\Sigma^e) - tr(\Sigma^r))^2$, $F_p(\xi) = W_c\,(tr(\Sigma^e) - tr(\Sigma^r))^2$ *as in [3], stability cannot be proved directly since the trace is a semi-norm with non-unique zero. Thus, we can only prove asymptotic stability to a set of equilibria. By imposing the constraints* $\bar{\Sigma}_0^e > 0$, $\bar{R}_k > 0$ *for all* $k \in \mathbb{Z}_{0+}$, *where* $\bar{\Sigma}^e$, $\bar{R}$ *are differences from setpoints,* $\bar{\Sigma}_k^e > 0$ *for every* $k \in \mathbb{Z}_{0+}$ *and stability can be proved using arguments from LaSalle's invariance principle.* ■

6 Case Study

For the ease of illustration, we show the behavior of PAC-MPC on a double integrator case study. A more realistic automated driving case study is described in details in [4]. We consider $x^s \in \mathbb{R}^2$, $u^s \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \end{bmatrix},\ E = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

in (17) with sampling period $T_s = 1$. Input $[u]_2 \in [0, 1]$ only impacts the environment measurement quality. The constraint sets in (2) are $\mathcal{X} = \{x \in \mathbb{R}^2 : \ |[x]_i| \le 10,\ i = 1, 2\}$, $\mathcal{U} = \{u \in \mathbb{R}^2 : , [u]_1 \in [-5, 5],\ [u]_2 \in [0, 1]\}$, and (17) is also subject to the ICCs

$$\mathbb{P}\Big[[x^s]_i - [x^e]_i \le 0\Big] \ge 1 - 0.05, \quad i = 1, 2, \qquad (27)$$

where $x^e \in \mathbb{R}^2$ is the environment state. We consider an environment with two elements, both of which are unknown but constant and measured subject to noise, so that the model is (19), with $A^e = I$, $B^e = 0$, $C^e = A^e$, $\zeta \sim \mathcal{N}(0, I)$, and

$$D^e_k = (1 - \beta[u_k]_2)\bar{D}, \qquad (28)$$

where $\beta \in (0, 1)$, $\bar{D} \in \mathbb{R}^{2\times 2}$, showing measurement dependency on $[u]_2$. The control objective is to regulate the system state to $r^x = 2$, with $r^u = 0$, and the environment estimate covariance to its steady-state for $x^s = r^x$, $u^s = r^u$. For weights in (24) assigned as $Q_c = \mathrm{diag}(0.1, 0.01)$, $R_c = \mathrm{diag}(0.1, 1)$, the controller is designed as described in Section 5, where the environment estimator and predictor are (20), (21), respectively, $L_k = -0.45 \cdot I$ for all $k \in \mathbb{Z}_{0+}$ in (20), (21), $N_p = 3$, $\rho = 3$, $S_c = 1$, $W_c = 2.5$ in (25). In (26), $\varrho = 2$, $M_s = 20 \cdot I_2$, P^c, K_f are obtained by solving (26d) as an LMI, and $\mathcal{Z}_f(\mathcal{M}^e)$ is the $\mathcal{O}_\infty$ set (23). Assumptions 1, 5, 6 are known to hold for the chosen system, cost and terminal set, Assumption 2 holds by the estimator design, Assumption 3 holds by choosing Σ^y based on the prediction error, and Assumption 4 holds due to the environment model where the disturbances and estimator error are zero-mean, see, e.g., [10].

Fig. 2 shows the closed-loop trajectories for different system initial states and the same initial environment estimate covariance. All the trajectories reach the desired steady-state while satisfying the constraints according to (27). Fig. 2 also shows that the closed-loop trajectories of the environment covariance has low sensitivity to the system initial state. Fig. 3 shows the closed-loop trajectories for fixed system initial state, but different environment estimate initial covariance. Also in this case, the trajectories stabilize to the setpoint.

Next, we consider a measurement quality dependent on the system state. That is, in (17) we substitute (28) by

$$[D^e_k(x^s_k, u^s_k)]_i = [D^e_0]_i(([\mu^e]_i - [x^s]_i)/\ell_i)^2, i = 1, 2, \qquad (29)$$

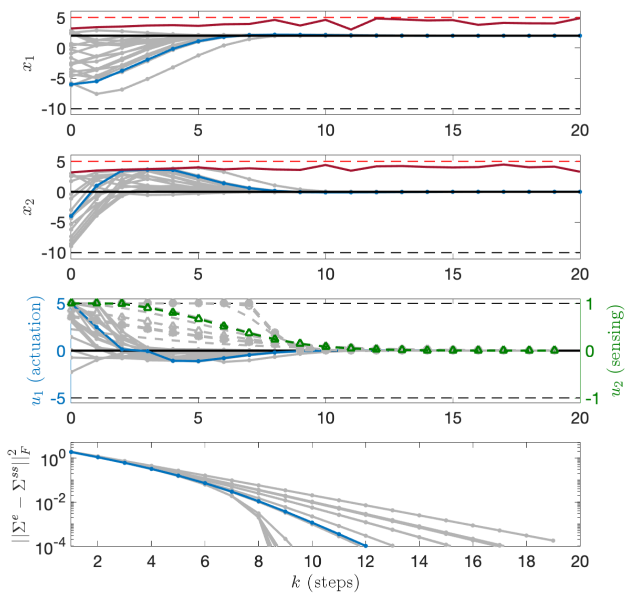

Fig. 2. PAC-MPC for double integrator with input-dependent measurements. Closed-loop state, input, and environment covariance trajectories for different initial system states and fixed initial environment uncertainty (gray), constraints (dash black), true environment constraints (dash red), perception input (dash gray), deterministic constraints (dash, black) . One simulation shown in blue, with corresponding ICCs based on environment estimate mean and covariance (solid, dark red) (solid, black).

where ℓ_i is a length-scale constant. Fig. 4 shows the closed-loop trajectories for different system initial states and the same initial environment estimate covariance. Also in this case, all the trajectories stabilize to the desired steady-state while satisfying the constraints most of the time according to (27), but with higher sensitivity to the initial environment estimate covariance.

To verify that the chance constraints provide the desired probability of satisfaction of constraints (27), we simulated 100 Monte Carlo runs, each of 200 time steps, when $r_x = 5.5$, i.e., slightly infeasible, to activate the constraints often and to get a tighter approximation of the empirical probability of constraint satisfaction. This resulted in a 98% constraint satisfaction, which is in agreement with the lower bound of 95% imposed by (27). The solution of the PAC-MPC optimal control problem (11) took in average 22 ms (and less than 30 ms in the worst case) at each control cycle when implemented in Matlab and solved with IPOPT via CasADi on a 2020 Intel MacBook Pro with 16 GB RAM. The reported computing times were obtained without any code or solver optimization.

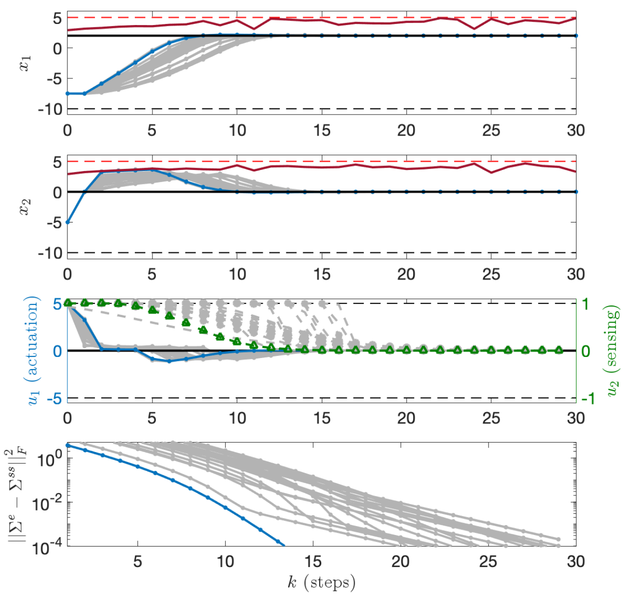

Fig. 3. PAC-MPC for double integrator with input-dependent measurements. Closed-loop state, input, and environment covariance trajectories for fixed system states and different initial environment uncertainty (gray), constraints (dash black), true environment constraints (dash red), perception input (dash gray), deterministic constraints (dash, black) . One simulation shown in blue, with corresponding ICCs based on environment estimate mean and covariance (solid, dark red) (solid, black).

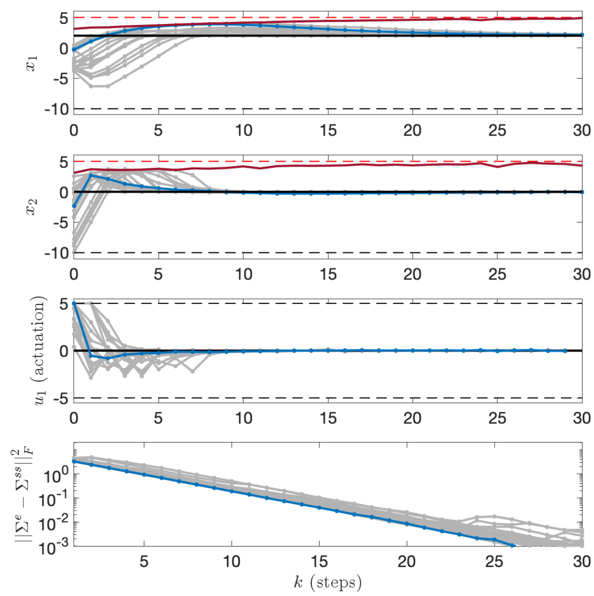

Fig. 4. PAC-MPC for double integrator with state-dependent measurements. Closed-loop state, input, and environment covariance trajectories for different initial system states and fixed initial environment uncertainty (gray), constraints (dash black), true environment constraints (dash red), deterministic constraints (dash, black). One simulation shown in blue, with corresponding ICCs based on environment estimate mean and covariance (solid, dark red) (solid, black).

7 Conclusions

We considered the control of a known system in an unknown environment that imposes constraints on the system operation. Since the environment is estimated via sensing, the constraints are uncertain. As the sensing quality is also affected by the system state and hence by the control actions, this results in an interdependence between sensing and control. We proposed a perception-aware chance-constrained model predictive control and a stabilizing design, which results in a constructive procedure, when the system dynamics are linear. Future work will involve exploiting sensing models constructed by machine learning that have already been tested in simulation but pose challenges to stability analysis, improving the computational efficiency using specialized solvers (e.g., [12]), and evaluating the proposed control strategy in applications using the corresponding sensing models, e.g., [27].

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